Simple and Robust HLLC Extensions of Two-Fluid AUSM for Multiphase Flow Computations

Keiichi Kitamura a, * and Taku Nonomura b

a) Nagoya University, 1 Furo-cho, Chikusa-ku, Nagoya, Aichi, 464-8603 Japan

b) Institute of Space and Astronautical Science, JAXA, 3-1-1 Yoshinodai, Sagamihara, Kanagawa, 252-5210 Japan

Abstract

A two-fluid AUSM+-up numerical flux function with the exact (Godunov) Riemann solver for the stratified flow model concept by C.-H. Chang, and M.-S. Liou [J. Comput. Phys. 225:840-873, 2007] has been extended for simple and robust computations of compressible multiphase flows. The present method replaces the Godunov part with the HLLC approximate Riemann solver with no-iteration procedure in a very simple manner: This two-fluid HLLC has been inspired by the work by X.Y., Hu, N.A., Adams, and G., Iaccarino [J. Comput. Phys. 228:6572-6589, 2009], but used in a totally different way. Numerical tests demonstrate that the present two-fluid AUSM+-up is, if only velocity and pressure in the middle zone are computed by HLLC, as robust as the original, Godunov-combined AUSM+-up, despite being free from iterations and convergence criteria.

Keywords: Two-fluid modeling, Multiphase flow, AUSM-family, HLLC, Stratified Flow

* corresponding author; tel: +81-52-789-3394, e-mail address: kitamura@fluid.nuae.nagoya-u.ac.jp
1. Introduction

Among many flux functions available today, a Harten-Lax-van_Leer-contact (HLLC) [1, 2], a Harten-Lax-van_Leer (HLL) type [3] approximate Riemann solver, is one of the popular methods because

1) It reasonably recognizes the middle zone ($\rho^*$ and $u^*$ in Fig. 1), including a contact discontinuity, separated by left and right running waves (accurate).

2) It requires no iterations (simple), as opposed to the Godunov’s exact Riemann solver.

3) It can be extended to low-speed flow problems by controlling dissipation [4].

The HLL-type schemes† have been used in many applications [5-9] and even extended to magneto-hydro-dynamics [10, 11]. Saurel and Abgrall [12] applied the two-wave HLL solver to their elaborate two-fluid approach and successfully simulated many problems involving various material interfaces. However, the two-wave HLL is known to have poor resolution of contact discontinuities (e.g., [13]). In addition, their method employed variables of three consecutive cells, and hence, its extension to unstructured grid does not appear to be straightforward. On the other hand, HLLC, the three-wave solver, is able to resolve interfaces economically (e.g., [13]), though many variants exist in the way the middle zone is expressed (as explained in [2]). However, applications of HLLC to the two-fluid multiphase flow simulations are very limited in literature, probably due to complexity of differencing different equations-of-state (EOSs) governing different phases which leads to further difficulty in specifying the middle zone variables. Among few examples, Shukla et al. [14] and Saurel et al. [15] each employed HLLC in a particular form from many possible formulations. However, their implementations are based on an ambiguously mixed fluid condition, similar to that of the single fluid computation code [16], and therefore they seem to be difficult to be applied to severe problems, e.g., a water-shock/air-bubble problem of $10^4$ pressure ratio which is much higher than $O(1)$ in the air-shock/helium-bubble tested in [15, 16]. Under these circumstances, this study will propose two simple implementations of HLLC (not the two-wave HLL) within the two-fluid, stratified flow [17] framework applicable to the water-shock/air-bubble problem.

† The HLL-type schemes include the original HLL, HLLE, HLLC, etc. The original HLL consists of two variants: two-wave and three-wave HLL solvers [3] (as claimed by Prof. Bram van Leer himself at “Four Decades of CFD: Looking Back and Moving Forward - A symposium celebrating the careers of Jameson, Roe and van Leer (JRV Symposium),” June 23, 2013, http://dept.ku.edu/~cfdku/JRV.html), although the former, along with HLLE in which the wave speeds are given by Einfeld, is more widely recognized as the ‘original’ HLL having no contact-resolving capability. The HLLC, restored this capability from the ‘two-wave’ HLL, is obviously the three-wave solver, but with many variants different in the way the contact (or middle zone between the left and right states) is expressed.
On the other hand, advection-upstream-splitting-method (AUSM) family schemes [18-21] are also widely used because of its robustness against shockwaves, and simplicity. Specifically, the latter allows us to straightforwardly extend the AUSM schemes to multiphase flows [17, 22-25]. Among them, the following two approaches should be noted, both falling into two-fluid modeling, based on the stratified flow model concept (Fig. 2) [17, 26].

1) AUSM\textsuperscript{+}\textendash up — AUSM-family standalone [22]: AUSM\textsuperscript{+}\textendash up [18] flux function, one of AUSM-family schemes [18-21], was successfully extended from the single-phase version without relying upon an expensive and exact Riemann solver [27]. By allowing difference of volume fractions $\alpha$ at a cell interface, we only need the flux of the same phase (i.e., gas-gas or liquid-liquid) (Figs. 3(a), (b)).

2) G-AUSM\textsuperscript{+}\textendash up — AUSM-family coupling with exact Riemann (Godunov) solver [17]: AUSM\textsuperscript{+}\textendash up, again, was used only to account for the subcell interface of the same phases, and the Godunov solver [27] was used for the portion of different phases. This approach utilizes iterations, as is well known, and can accurately predict which of the phases will penetrate into the other at the next time step (i.e., whether the gas flows into the liquid region, or vice versa, at the gas-liquid interface) (Figs. 3(a)-(d)).

It is found, from the recent study by one of the authors and coworkers [23], that AUSM\textsuperscript{+}\textendash up is more efficient, but less robust than G-AUSM\textsuperscript{+}up in general. In a severe problem where a large pressure jump of, say, 1000 times, is present at a phase discontinuity, AUSM\textsuperscript{+}\textendash up suffers from negative pressure in the middle of the two phases; G-AUSM\textsuperscript{+}\textendash up can remedy it, but many iterations are called for in the Godunov solver (if its convergence criterion is strict) - then, what if HLLC solver is used instead? This is the motivation of the present work, and our natural expectation is that HLLC-coupled AUSM\textsuperscript{+}\textendash up (denoted as “H-AUSM\textsuperscript{+}\textendash up” here) is as robust as G-AUSM\textsuperscript{+}\textendash up, and nearly as simple as AUSM\textsuperscript{+}\textendash up. Although the HLLC has several variants [2], it is common to use Roe-averaged variables in wave speed estimates as done, for instance, by Hu et al. [28] which inspired the present work in deriving a two-fluid version of HLLC. In addition, although we chose AUSM\textsuperscript{+}\textendash up as an AUSM-family flux here, other two-fluid fluxes [23] such as SLAU [19], SLAU2 [20], and AUSMPW+ [21] can be extended in the same manner (then, the robustness of the original flux will be taken over).

It should be noted that more sophisticated methods such as sharp-interface-method combined with adaptive mesh refinement and inclusion of viscous effects [29], or the latest front-tracking method [30], have achieved high-resolution of a bubble interface interacting with a shock very recently. The two-fluid approach, on the other hand, still has advantage in resolving multiple bubbles efficiently in an averaged manner. Further, the present framework
can be combined with a sharpening technique [31, 32] for a better resolution of a single bubble, and such a work is ongoing [33].

The paper is organized as follows. In Sec. 2, numerical methods including two-fluid AUSM\(^+\)-up, G-AUSM\(^+\)-up, and the proposed H-AUSM\(^+\)-up are explained. Then in Sec. 3 the H-AUSM\(^+\)-up in two ways of implementations are tested in benchmark test and their performances are demonstrated. Section 4 will summarize the current work.

2. Numerical Methods

2.1 Two-Fluid Modeling (or Effective-Fluid Modeling, EFM)

The 2D compressible Euler equations in two-fluid modeling are written as:

\[
\frac{\partial \mathbf{Q}_k}{\partial t} + \frac{\partial \mathbf{E}_k}{\partial x} + \frac{\partial \mathbf{F}_k}{\partial y} = \mathbf{P}_k^{\text{int}}, \quad k = 1, 2, \tag{1a}
\]

\[
\begin{align*}
\mathbf{Q}_k &= \begin{bmatrix} \alpha \rho \\ \alpha p u \\ \alpha \rho v \\ \alpha \rho E \end{bmatrix}, \\
\mathbf{E}_k &= \begin{bmatrix} \alpha p u \\ \alpha p u^2 + \alpha p \\ \alpha p u v \\ \alpha \rho u H \end{bmatrix}, \\
\mathbf{F}_k &= \begin{bmatrix} \alpha \rho v \\ \alpha p v u \\ \alpha p v^2 + \alpha p \\ \alpha \rho v H \end{bmatrix}, \\
\mathbf{P}_k^{\text{int}} &= \begin{bmatrix} 0 \\ p_{k}^{\text{int}} \frac{\partial \alpha}{\partial x} \\ p_{k}^{\text{int}} \frac{\partial \alpha}{\partial y} \\ -p_{k}^{\text{int}} \frac{\partial \alpha}{\partial t} \end{bmatrix}, \tag{1b}
\end{align*}
\]

\[\alpha_g + \alpha_l = 1, \tag{2}\]

\[p_g = p_l \equiv p, \tag{3}\]

\[p_{g}^{\text{int}} = p_{l}^{\text{int}} \equiv p^{\text{int}}, \tag{4}\]

\[p^{\text{int}} = p - \delta p^*, \tag{5}\]

where \(\alpha\) is the volume fraction of a fluid, \(\rho\) the density, \(u\) and \(v\) are the velocity components in Cartesian coordinates, \(E\) is the total energy per unit mass \([E = e + (p/\rho), e\) being the internal energy\], \(p\) is the pressure, and \(H\) is the total enthalpy \([H = E + (p/\rho)]\). Since we treat only gas-liquid systems in this study, \(k = 1, 2\) is interchangeable with \(k = g, l\), where \(g\) represents gas and \(l\) represents liquid. As in single-fluid equations, \(\mathbf{Q}\) is the conservative-variable vector; \(\mathbf{E}\) and \(\mathbf{F}\) are the inviscid flux vectors in the \(x\)- and \(y\)-directions, respectively, but with \(\alpha\) included; \(p_{k}^{\text{int}}\) is the so-called interface pressure. The Eq. (2) expresses the compatibility relation for volume fractions, Eq. (3) and Eq. (4) assume the pressure equilibrium, and Eq. (5) gives the interface pressure, \(p^{\text{int}}\), as a departure from \(p\) by \(\delta p^*\), which will be
explained further in Sec. 2-3. Now we have 14 unknowns \([\alpha, \rho, u, v, e, p, p^{\text{int}}]\), closed by 12 equations [Eqs. (1)–(5)] with two EOSs described later in Sec. 2-4.

**2.2 Stratified Flow Model and Discretization**

Within the framework of a finite-volume, shock-capturing method, we follow the concept of stratified flow model, which was proposed first by Stewart and Wendroff [26] (Fig. 2(a)) and later refined by Chang and Liou [17] and Liou et al. [22], for constructing a discrete model consistent with the continuous equations, Eq. (1). Hence, it is clear to interpret that the interfacial pressure \(p^{\text{int}}\) must work only within each computation cell, and the volume fractions are assumed to be continuous within the cell but are allowed to have a jump at the cell boundaries (Fig. 2(b)). The discretized form of Eq. (1), by retaining only the 1D form for illustration, is expressed as:

\[
\frac{V_j}{\Delta t} \Delta Q_j + E_{j+1/2} S_{j+1/2} - E_{j-1/2} S_{j-1/2} = p^{\text{int}}_j \begin{bmatrix} 0 \\ \alpha_{j+1/2, \text{L}} - \alpha_{j-1/2, \text{R}} \\ V_j \left( \alpha_j^{n+1} - \alpha_j^n \right) / \Delta t \end{bmatrix} + S_j,
\]

where the phase-subscript \(k\) is omitted, and \(j\) is the cell index, \(V_j\) is the volume of cell \(j\), and \(S_{j+1/2}\) is the area of the interface between cells \(j\) and \(j+1\). All the cell-interface variables (such as \(\alpha_{j+1/2, \text{L}}\)) are calculated by the spatially second-order accurate MUSCL interpolation [34] with Van Albada’s limiter [35] (a limiter coefficient is set as \(10^{-20}\); limiter effects are surveyed in [23]). A three-stage, third-order total-variation-diminishing (TVD) Runge-Kutta method [36] is used for time integration, and its details will be explained later in Sec. 2-6. The numerical code had been extended [23] from a single-fluid version previously used by Kitamura et al. [37].

**2.3 Interface Pressure**

The interface pressure, \(p^{\text{int}}\), introduced by Stuhmiller [38], working at a phase interface within a cell according to Liou et al. (Fig. 2(b)) [22], is expressed as:

\[
p^{\text{int}} = p - \delta p^* ,
\]

and for a gas-liquid system, \(\delta p^*\) is usually given by

\[
\delta p^* = \sigma \frac{\alpha_g \rho_1 \rho_g \rho_l}{\alpha_g \rho_l + \alpha_l \rho_g} \left| u_l - u_g \right|^2 ,
\]
or after assuming $\rho_l >> \rho_g$ and $(\alpha_l, \alpha_g)$ are finite, it is simply

$$
\dot{\phi}^* = C_p^* \alpha_l \rho_g |u_i - u_g|^2.
$$

(8)

The interface pressure coefficient, $C_p^*$ or $\sigma$, should be large enough (at least larger than or equal to unity) to keep the system hyperbolic [22, 39]. We will make use of the simplified form in Eq. (8) with $C_p^* = 2.0$ (a discussion about the effect of this and other values is given in [22]).

Furthermore, in order to prevent $\rho^{int}$ from deviating too much from the static pressure $\rho$, a limit should be imposed so that $\dot{\phi}^*$ does not exceed a fraction of $\rho$:

$$
\dot{\phi}^* = \min(\dot{\phi}^*, \epsilon_p \rho).
$$

(9)

where the value of $\epsilon_p = 0.01$ suggested in [17] is also adopted for all the numerical tests here.

2.4 Equation of State (EOS)

For closure of the system we adopted the stiffened-gas model proposed by Harlow and Amsden [40] to represent the EOS:

$$
p_k = \rho_k \frac{\gamma_k - 1}{\gamma_k} C_{pk} T_k - p_{k\infty},
$$

(10a)

$$
e_k = \frac{C_{pk} T_k}{\gamma_k} + \frac{p_{k\infty}}{\rho_k},
$$

(10b)

$$
a_k = \left( \frac{\gamma_k (p_k + p_{k\infty})}{\rho_k} \right)^{\frac{1}{2}},
$$

(10c)

where $e_k$ is the internal energy per unit mass of fluid $k$ and $a_k$ is the speed of sound. It is noted that the standard ideal gas becomes a subset of the stiffened gas, hence it is used to describe both gas and liquid states only with different parameter values [22]:

$$
\gamma_g = 1.4, \quad C_{pg} = 1004.5 \, [\text{J/(kg K)}], \quad p_{g\infty} = 0 \, [\text{Pa}] \quad \text{for gas},
$$

(11a)

$$
\gamma_l = 2.8, \quad C_{pl} = 4186 \, [\text{J/(kg K)}], \quad p_{l\infty} = 8.5 \times 10^8 \, [\text{Pa}] \quad \text{for liquid}.
$$

(11b)
It is reported by Jolgam et al. in [41] that although the stiffened-gas EOS is relatively simple, it has almost the same accuracy as more sophisticated Tait’s (for water) or van der Waals’s EOS (for air) in several benchmark tests.

2.5 Numerical Fluxes

2.5.1 AUSM+-up

AUSM+-up by Liou [18], one of AUSM-family flux functions, extended for two-fluid multiphase flow computations [22], is used to calculate inviscid numerical fluxes at cell interfaces for each phase, denoted as $F_{k,1/2,L/R}$, where $L$ and $R$ indicate left and right cells, respectively. The numerical flux is expressed as:

$$F_{k,1/2,L/R} = \frac{\dot{m}_{k,1/2} + \dot{m}_{k,1/2}}{2} \Psi_{k,L} + \frac{\dot{m}_{k,1/2} - \dot{m}_{k,1/2}}{2} \Psi_{k,R} + \alpha_{k,1/2,L/R} \widetilde{p}_{k,1/2} \mathbf{N}$$ (12a)

$$\Psi_k = (\alpha, \alpha u, \alpha v, \alpha H)^T, \quad \mathbf{N} = \left(0, n_x, n_y, 0\right)^T.$$ (12b)

The mass flux is given by

$$\dot{m}_{k,1/2} = M_{k,1/2} a_{1/2} \begin{cases} \rho_{k,L} & \text{if } M_{k,1/2} > 0 \\ \rho_{k,R} & \text{otherwise} \end{cases}$$ (13a)

$$M_{k,1/2} = M_{(4)k} \begin{pmatrix} M_{k,L} \end{pmatrix}_{1/2} + M_{(4)k} \begin{pmatrix} M_{k,R} \end{pmatrix}_{1/2} + M_{pk},$$ (13b)

$$M_{(4)}^+ (M)_{1/2} = \begin{cases} \frac{1}{2} (M \pm |M|), & \text{if } |M| \geq 1 \\ \pm \frac{1}{4} (M \pm 1)^2 \pm \frac{1}{8} (M^2 - 1)^2, & \text{otherwise} \end{cases}$$ (13c)

$$M_{pk} = \frac{K_p}{f_a} \max \left(1 - \bar{M}_k^2, 0\right) \frac{p_L - p_k}{\bar{p}_k a_{1/2}^2}, \quad \bar{p}_k = \frac{\rho_{k,L} + \rho_{k,R}}{2},$$ (13d)

$$M_k = \frac{V_{k,a}}{a_{1/2}} = \frac{u_x n_x + v_y n_y}{a_{1/2}},$$ (13e)

$$\bar{M}_k^2 = \frac{V_{k,a}^2 + V_{k,a}^2}{2 a_{1/2}^2},$$ (13f)
where \( f_a = 1 \) here (hence, no prescribed Mach number is used), and the speed of sound, \( a_{1/2} \), common to gas and liquid [22], is

\[
a_{\frac{1}{2}} = \frac{1}{2} \left( a_{k,\frac{1}{2}} + a_{e,\frac{1}{2}} \right),
\]

(13g)

\[
a_{k,\frac{1}{2}} = \frac{a_{k,L} + a_{k,R}}{2},
\]

(13h)

where the arithmetic mean of the left and right states is used for the speed of sound for each fluid, \( a_{k,1/2} \). It had been confirmed in [23] that the present choice gives robust performances in general.

Then, the pressure flux is

\[
\tilde{p}_{k,\frac{1}{2}} = P_{(i \leftrightarrow k)}^+(M_{k,L}) \frac{1}{\rho_{L}} p_{L} + P_{(i \leftrightarrow k)}^-(M_{k,R}) \frac{1}{\rho_{R}} p_{R} + p_{ad},
\]

(13i)

\[
P_{(i \leftrightarrow k)}^+(M) \frac{1}{\rho_{L}} \begin{cases} 
\frac{1}{2} (1 \pm \text{sign}(M)), & \text{if } |M| \geq 1 \\
\frac{1}{4} (M \pm 1)^2 (2 + M) \pm \frac{3}{16} M (M^2 - 1)^2, & \text{otherwise}
\end{cases}
\]

(13j)

\[
p_{ad} = -K_n P_{(i \leftrightarrow k)}^+(M_{k,L}) \frac{1}{\rho_{L}} \left| \frac{1}{\rho_{R}} P_{(i \leftrightarrow k)}^-(M_{k,R}) \right| \tilde{f}_{k} f_{a,\frac{1}{2}} (V_{k,n} - V_{k,n+}),
\]

(13k)

where tunable parameters are \( K_p \) and \( K_n \), both set as unity as in the previous works [17, 22] for stability reason [23].

### 2.5.2 G-AUSM\(^+-\)up: Combination with Godunov Riemann Solver

As stated in Introduction, making use of the Godunov Riemann solver is another approach (denoted as “G-AUSM\(^+-\)up”). In practice, the Godunov solver is used only when the void fraction jump (or the “effective length,” \( \Delta_{\text{eff}} = |a_{g,1/2,L} - a_{g,1/2,R}| \)) is larger than the prescribed threshold, or \( \varepsilon \) (Fig. 3(c)); otherwise, only an AUSM\(^+-\)up flux is used (Fig. 3(b)) [20]. In other words, G-AUSM\(^+-\)up includes AUSM\(^+-\)up in cases of small effective length (Fig. 3(d)). These two methods are illustrated in Fig. 3: In either case, at least AUSM\(^+-\)up flux is used.

- if \( \Delta_{\text{eff}} = |a_{g,1/2,L} - a_{g,1/2,R}| < \varepsilon \): an AUSM-family flux is used everywhere;

- otherwise: AUSM-family is used for gas-gas and liquid-liquid interfaces, and Godunov is used elsewhere
The Godunov Riemann solver [27] for stiffened-gas EOS is written in a very similar manner as in the ideal gas [2, 17]. For instance, if the liquid and gas phases are at the left and right sides, respectively (i.e., $\alpha_{g,1/2,L} + \varepsilon < \alpha_{g,1/2,R}$), as in Fig. 3(c), or in Fig. 1 of $x$-$t$ diagram (a more common way to illustrate a Riemann problem),

1) First, the middle zone pressure $p^*$ is estimated, such as $p^* = 0.5(p_l + p_g)$

2) If $p^* > p_l$, the left-running wave “$u - a$” is assumed to be a shock wave, and hence, the velocity jump $\Delta u_L$ across it is given by

$$\Delta u_L = (p^* - p_l) \left[ \frac{2}{\rho_l \left( (\gamma_l + 1) \left[ p^* + p_{le} \right] + (\gamma_l - 1) \left[ p_l + p_{le} \right] \right)} \right]^{1/2}, \quad (14a)$$

Otherwise, the expansion waves should be there, resulting in the velocity difference $\Delta u_R$ by the isentropic relation:

$$\Delta u_L = \frac{2a_l}{\gamma_l - 1} \left[ \left( \frac{\xi_l}{\gamma_l} \right)^{\frac{\gamma_l - 1}{\gamma_l}} - 1 \right], \quad (14b)$$

with

$$\xi_l = \frac{p^* + p_{le}}{p_l + p_{le}}, \quad (14c)$$

3) Similarly, if $p^* > p_R$, the right-running wave “$u + a$” is a shock, and the velocity jump $\Delta u_R$ is:

$$\Delta u_R = (p^* - p_g) \left[ \frac{2}{\rho_g \left( (\gamma_g + 1) p^* + (\gamma_g - 1) p_g \right)} \right]^{1/2}, \quad (14d)$$

Otherwise,

$$\Delta u_R = \frac{2a_g}{\gamma_g - 1} \left[ \left( \frac{\xi_g}{\gamma_g} \right)^{\frac{\gamma_g - 1}{\gamma_g}} - 1 \right], \quad (14e)$$

where

$$\xi_g = \frac{p^*}{p_g}. \quad (14f)$$

4) Since $u^* = u_l - \Delta u_L$ and $u^* = u_g + \Delta u_R$, the following should be satisfied:

$$f^* = u_g + \Delta u_R - u_l + \Delta u_L = 0, \quad (14g)$$
If $|f^*|$ is above the specified tolerance (default: $10^{-4}$, in this study), the middle zone pressure $p^*$ is updated as:

$$
(p^*)^{m+1} = (p^*)^m - \frac{f^*}{\partial p^*/\partial (p^*)} = (p^*)^m - \frac{f^*}{\partial (\Delta u_g)/\partial (p^*)} + \partial (\Delta u_L)/\partial (p^*)
$$

where $m$ is the Newton iteration stage, and the steps 2) – 4) are repeated until the tolerance is satisfied.

5) Once the $f^*$ is obtained, the other middle zone variables are calculated as:

$$u^* = 0.5(u_g + \Delta u_g + u_l - \Delta u_l),
$$

$$
\rho_i^* = \begin{cases}
\rho_i \left( \frac{\theta_i + \xi_i}{\gamma - 1} \right) & \text{if } p^* > p_i \\
\rho_i \left( \xi_i \right)^{\gamma_i} & \text{otherwise}
\end{cases}
$$

$$
\rho_g^* = \begin{cases}
\rho_g \left( \frac{\theta_g + \xi_g}{\gamma_g - 1} \right) & \text{if } p^* > p_g \\
\rho_g \left( \xi_g \right)^{\gamma_g} & \text{otherwise}
\end{cases}
$$

with

$$
\theta_k = \frac{\gamma_k + 1}{\gamma_k - 1}
$$

6) If $u^* > 0$, the liquid (left) flows into the gas (right) at the interface (Fig. 1(a)), and thus, the liquid flux is given as:

$$F_{t-g,1/2-Godunov} = \Delta_{eff} \cdot \left[ \rho_i u^* \rho_i u^* \rho_i v_i \rho_i u^* h_i^* \right],
$$

where

$$h_i^* = \frac{\gamma_i}{\gamma_i - 1} \frac{p_i^* + p_{i,sc}}{\rho_i} + \frac{1}{2} \left( u_i^{*2} + v_i^{*2} \right),
$$

and otherwise, the gas (right) phase comes into the liquid (left) (Fig. 1(b)) as

$$F_{t-g,1/2-Godunov} = \Delta_{eff} \cdot \left[ \rho_g u^* \rho_g u^* \rho_g v_g \rho_g u^* h_g^* \right],
$$
where

\[ h_g^* = \frac{\gamma_g}{\gamma_g - 1} \frac{p_g^*}{\rho_g^*} + \frac{1}{2} \left( u_g^2 + v_g^2 \right). \]  

(14p)

7) Furthermore, interface pressure flux is also applied as in Eq. (7),

\[ p_{\text{int}} = p_g^* - \sigma \frac{\alpha_{\text{in}}^g \alpha_{\text{in}}^l \rho_l}{\alpha_{\text{in}}^g \rho_g + \alpha_{\text{in}}^l \rho_l} |\mathbf{u}_l - \mathbf{u}_g|^2, \]  

(14q)

where \( \sigma = 2.0 \), the Eq. (9) is again applied to the second term at the right hand side, and

\[ \alpha_{\text{in}}^g = 0.5(\alpha_{g,1/2,L} + \alpha_{g,1/2,R}), \quad \alpha_{\text{in}}^l = 1 - \alpha_{\text{in}}^g \]  

(14r)

Again, the Godunov solver is used at gas (left)-liquid (right) or liquid (left)-gas (right) interfaces only; gas-gas and liquid-liquid interfaces are treated by AUSM+up. Thus, the flux of the AUSM+up portion is calculated according to Eq. (12), but with the modification of \( \alpha_{k,1/2,R} \) to \( \min(\alpha_{k,1/2,L}, \alpha_{k,1/2,R}) \) so that the common and minimum void fraction of both sides is used and the room for Godunov solver \( \Delta_{\text{eff}} = |\alpha_{g,1/2,L} - \alpha_{g,1/2,R}| \) is left, in contrast to the AUSM+up standalone (Sec. 2.5.1). Therefore, the resulting flux (G-AUSM+up) is expressed as summation of the gas-gas part (AUSM+up), the liquid-liquid part (AUSM+up), and the liquid-gas part (Godunov) (again, if \( \alpha_{G,1/2,L} + \varepsilon < \alpha_{g,1/2,R} \)), as illustrated in Fig. 3(c).

\[
\begin{align*}
F_{L,1/2,L} & = F_{L,1/2,\text{AUSM-family}} + F_{L,1/2,\text{Godunov}} + \Delta_{\text{eff}} \cdot p_{\text{int}} \\
F_{g,1/2,L} & = F_{g,1/2,\text{AUSM-family}} \\
F_{L,1/2,R} & = F_{L,1/2,\text{AUSM-family}} + F_{L,1/2,\text{Godunov}} + \Delta_{\text{eff}} \cdot p_{\text{int}} \\
F_{g,1/2,R} & = F_{g,1/2,\text{AUSM-family}} \\
F_{L,1/2,L} & = F_{L,1/2,\text{AUSM-family}} + \Delta_{\text{eff}} \cdot p_{\text{int}} \\
F_{g,1/2,L} & = F_{g,1/2,\text{AUSM-family}} + F_{L,1/2,\text{Godunov}} \\
F_{L,1/2,R} & = F_{L,1/2,\text{AUSM-family}} + F_{L,1/2,\text{Godunov}} + \Delta_{\text{eff}} \cdot p_{\text{int}} \\
F_{g,1/2,R} & = F_{g,1/2,\text{AUSM-family}} \\
\end{align*}
\]  

(14s)

In the case of a gas (left)-liquid (right) interface, \( L \) and \( R \) are flipped over. In addition, for the AUSM+up in this method, Eq. (13g) for the gas-liquid-averaged acoustic speed is not used for AUSM+up because each phase is computed totally independently.

2.5.3 H-AUSM+up: Combination with HLLC Riemann Solver (Present)
Although G-AUSM⁺-up was shown to be more robust than AUSM⁺-up [23], its obvious drawback lies in the huge computational cost arising from many iterations (Eqs.(14a)-(14i)) when the middle zone velocity \( u^* \) and pressure \( p^* \) are to be calculated, particularly when a large pressure jump is present and AUSM⁺-up standalone fails.

Inspired by Hu, Adams, and Iaccarino [28], we try to determine these \( u^* \) and \( p^* \) but from only a single step of the simpler and faster HLLC Riemann solver without calling for iterations. The HLLC-based procedure for computing \( u^* \) and \( p^* \), rather than Godunov iterations (Eqs.(14a)-(14i)), is explained as follows:

1) First, the Roe-averaged velocity and acoustic speed is calculated according to [28, 42]

\[
\hat{a}^2 = \hat{\gamma} \left( \frac{\hat{p}}{\hat{\rho}} \right),
\]

where

\[
\hat{\gamma} = \mu(\gamma),
\]

\[
\left( \frac{\hat{p}}{\hat{\rho}} \right) = \frac{\hat{\gamma} - 1}{\hat{\gamma}} \left[ \hat{h} - \frac{\hat{u}^2}{2} \right]
\]

\[
= \frac{\hat{\gamma} - 1}{\hat{\gamma}} \left[ \mu \left( \frac{p + p_\pm}{\rho} \right) + \frac{1}{2} \sqrt{\rho_\pm} \rho_\mp \left( u_i - u_\pm \right)^2 \right],
\]

\[
\mu(\gamma) = \sqrt{\rho_\pm f_\gamma + \rho_\pm f_\gamma},
\]

and

\[
\hat{u} = \mu(u).
\]

Note that unlike Morin et al. [43], the void fraction \( \alpha \) should not be included in the Roe-average in Eq.(15d), because the Riemann solver in this study is called only at the liquid-gas (or gas-liquid) interface portion (i.e., one side is already filled by the liquid, and the other by the gas) (Fig. 1). Note also that such a phase-interface treatment has not been established yet for AUSM-family schemes that require information as to which of the liquid or the gas flows into the other \textit{a priori}. On the other hand, at gas-gas and, in particular, liquid-liquid interfaces, AUSM-family fluxes are still favorable because they are simple and have been well-validated at low speeds [18, 20, 22].

2) Then, the middle zone velocity \( u^* \) and pressure \( p^* \) are readily obtained through the HLLC procedure [28].
In order to avoid negative $p^*$, the following limit is also imposed.

$$p^* = \max \left[ p^*, \varepsilon_{p^*} \cdot \min \left( p_L, p_R \right) \right],$$

where $\varepsilon_{p^*} = 1.e-6$. Using those $u^*$ and $p^*$, the same procedure in G-AUSM$^+$-up (without iterations) from Eq. (14j) is continued. This method is denoted as “H-AUSM$^+$-up” in the rest of the paper.

### 2.5.4 H-AUSM$^+$-up (2): Combination with HLLC Riemann Solver (Variant)

As another option, all the variables, rather than only $u^*$ and $p^*$, can be calculated according to HLLC [2]. In this implementation, once the $u^*$ and $p^*$ are obtained by Eqs. (15f) and (15g):

3) If $u^* > 0$, the liquid (left) flows into the gas (right) at the interface (Fig. 1(a)), and thus, the liquid flux is:

$$\mathbf{F}_{l-g,1/2-HLLC} = \Delta_{eff} \left[ \begin{array}{c} \frac{\rho_l u_i}{u^*} \\ \frac{\rho_l u_i^2}{\rho_l u_i v_i} \\ e_i + p_i \\ \frac{b_l \cdot \rho_i}{b_l - u_i} \end{array} \right] + b_L \cdot \rho_i \left( \frac{b_l - u_i}{b_l - u^*} \right) \left[ \begin{array}{c} 1 \\ u^* \\ v_i \\ e_i \end{array} \right],$$

$$\left( \frac{e_k}{\rho_k} \right)_{HLLC} = \left( \frac{e_k}{\rho_k} \right) + \left( u^* - u_k \right) \left[ u^* + \frac{p_k}{\rho_k (b_l - u_k)} \right],$$

and otherwise, the gas (right) phase comes into the liquid (left) (Fig. 1(b)) as

$$\mathbf{F}_{g-l,1/2-HLLC} = \Delta_{eff} \left[ \begin{array}{c} \frac{\rho_g u_g}{u^*} \\ \frac{\rho_g u_g^2}{\rho_g u_g v_g} \\ e_g + p_g \\ \frac{b_g \cdot \rho_g}{b_g - u_g} \end{array} \right] + b_R \cdot \rho_g \left( \frac{b_g - u_g}{b_g - u^*} \right) \left[ \begin{array}{c} 1 \\ u^* \\ v_g \\ e_g \end{array} \right],$$

Then the pressure flux is separately calculated as in Eq.(14q) as in G-AUSM$^+$-up, and the final flux has the same form as in Eq.(14s) only with the replacement of “$\mathbf{F}_{l-g,1/2-Godunov}$” with “$\mathbf{F}_{l-g,1/2-HLLC}$”. This version is called “H-AUSM$^+$-up (2)” in this paper. Note that both H-AUSM$^+$-up and H-AUSM$^+$-up (2) are economical compared with G-
AUSM⁺-up, because of no iterations involved in the flux computation procedure. Flux evaluations for all the schemes tested in this note are summarized in Table 1.

2.6 Time Integration, Decoding, and Update of Variables

Equation (6) is rewritten in the three-stage TVD Runge-Kutta [36] form as:

\[
\hat{Q}_j^{(0)} = \hat{Q}_j^* + \frac{\Delta t}{V_j} R_j^n,
\]

\[
\hat{Q}_j^{(2)} = \frac{3}{4} \hat{Q}_j^* + \frac{1}{4} \hat{Q}_j^{(0)} + \frac{1}{4} \frac{\Delta t}{V_j} R_j^{(1)},
\]

\[
\hat{Q}_j^{n+1} = \frac{1}{3} \hat{Q}_j^* + \frac{2}{3} \hat{Q}_j^{(2)} + \frac{2}{3} \frac{\Delta t}{V_j} R_j^{(2)},
\]

\[
\hat{Q}_j = [0 \quad 0 \quad p_{int}^n \alpha_j] = [\hat{Q}_1 \quad \hat{Q}_2 \quad \hat{Q}_3],
\]

\[
R_j = -[E_{j+1/22} S_{j+1/2} - E_{j-1/22} S_{j-1/2}] + \left[ p_{int}^n \left( \alpha_{j+1/2_L} - \alpha_{j-1/2_R} \right) \right] + S_j,
\]

where \( k \) is omitted, and the term \( (\rho^{int} \alpha) \) is included in \( \hat{Q} \) as in Eq. (17d) [22, 25], but \( p^{int} \) is frozen at the \( n \)th time step value throughout the Runge-Kutta stages [25].

Once \( \hat{Q}^{n+1} \) is obtained, the following decoding process [17, 22] is required to update \( p^{n+1} \) and \( \alpha_k^{n+1} \) by solving \( p^{n+1} \) from

\[
F(p) = p^2 - Bp - C = 0.
\]

Since \( p \) is positive, a unique root is determined:

\[
p = \frac{1}{2} \left( B + \sqrt{B^2 + 4C} \right).
\]

The volume fraction follows:

\[
\alpha_k = \frac{A_k}{p + \hat{a}_k},
\]

where
Since a huge value of \( p_{\text{liq}} \) is involved in Eqs. (17) and (19), the resultant numerical errors can be large. Thus, a Newton iteration method is introduced to improve accuracy by solving Eq. (19b) simultaneously for both the liquid and gas phases [17]:

\[
\begin{align*}
\dot{A}_k &= (\gamma_k - 1) \left( \dot{Q}_{3,k} - \frac{\dot{Q}_{2,k}^2}{2\dot{Q}_{1,k}} \right), \\
B &= \sum_{k=1}^{2} \left( \dot{A}_k - \dot{\alpha}_k \right), \\
C &= \dot{\alpha}_1 \dot{A}_2 + \dot{\alpha}_2 \dot{A}_1 - \dot{\alpha}_1 \dot{\alpha}_2, \\
\dot{\alpha}_k &= \gamma_k p_{k,\text{liq}} + (\gamma_k - 1) p_{\text{int}}.
\end{align*}
\]

(19c)

(19d)

(19e)

(19f)

Since a huge value of \( p_{\text{liq}} \) is involved in Eqs. (17) and (19), the resultant numerical errors can be large. Thus, a Newton iteration method is introduced to improve accuracy by solving Eq. (19b) simultaneously for both the liquid and gas phases [17]:

\[
\begin{align*}
F_g &= \left( p + \dot{\alpha}_g \right) \alpha_g - \dot{A}_g = 0, \\
F_l &= \left( p + \dot{\alpha}_l \right) \alpha_l - \dot{A}_l = 0.
\end{align*}
\]

(20)

Usually, a few iterations are enough to drive pressure error below \( 10^{-5} \) [Pa].

Then, following Paillère et al. [24] and Chang and Liou [17], variables of the “vanishing” phase (i.e., \( \epsilon_{\text{min}} \leq \alpha_1 \leq \epsilon_{\text{max}} \)) are blended with those of the remaining phase (i.e., \( \alpha_2 \approx 1 \)) to enhance stability:

\[
(q_1)_{\text{adj}} = G(\xi_1)q_1 + (1 - G(\xi_1))q_2, \quad q = u, T,
\]

(21a)

\[
G(\xi_1) = -\xi_1^2 (2\xi_1 - 3),
\]

(21b)

\[
\xi_1 = \frac{\alpha_1 - \epsilon_{\text{min}}}{\epsilon_{\text{max}} - \epsilon_{\text{min}}},
\]

(21c)

where \( G \) is a smooth function satisfying \( G(0) = 0, G(1) = 1, \) and \( G'(0) = G'(1) = 0. \) The small values of \( \epsilon_{\text{min}} \) and \( \epsilon_{\text{max}} \) are chosen as \( 0.1 \epsilon \left(=10^{-8}\right) \) and \( 10^7 \epsilon \left(=10^{-4}\right) \) in this paper, if not mentioned otherwise. If \( \alpha_1 \) is below \( \epsilon_{\text{min}} \), \( \alpha_1 = \epsilon_{\text{min}} \) is enforced. Here \( k = 1, 2 \) is interchangeable with \( k = g, l \) and \( k = l, g \) both.

### 2.7 Boundary Conditions

Since a cell-centered, 2D structured grid solver is used here, the following typical boundary conditions using the typical “ghost cell” approach are applied as in [24]:

- **Inlet:** all the variables are imposed except for pressure, which is extrapolated from the interior cell.
- **Outlet:** only pressure is imposed, and all the other variables are extrapolated from the interior cell.
• Side (for a 1D problem): all the variables are extrapolated from the interior cell (for the direction irrelevant to the problem to be solved).

• Slip: also known as “mirror” condition, in which the opposite sign is put to the velocity component normal to the boundary, and all the other variables are extrapolated from the interior cell.

3. Numerical Examples

We will demonstrate that the currently proposed H-AUSM+-up (in two ways for implementations) is successfully used for multiphase flows in benchmark problems as well as G-AUSM+-up which calls for iterations. The computations are conducted by H-AUSM+-up and H-AUSM+-up (2), and compared with (robust but expensive) G-AUSM+-up and (cheap but less robust) AUSM+-up results for reference.

As in the previous work [23], the following “CFL-like number,” taken between 0.05 and 0.2, is used to determine the time step:

\[
CFL = \Delta t \left( \min_j \left( \frac{\Delta x}{\max(a_j, d_j) + \max(\|u_j\|, \|u_l\|)} \right) \right). \tag{22}
\]

Note that values of \(\varepsilon\), \(\varepsilon_{\text{min}}\), and \(\varepsilon_{\text{max}}\) also will be set differently depending on problems, following the precedent work [17, 22, 23].

3.1 Moving Phase Contact Discontinuity

A moving phase contact discontinuity between air and water separated at \(x = 5\) m [17] is solved first. It is desired to accurately capture this phase discontinuity, across which pressure equilibrium should be maintained. A grid of 200 uniform cells is used for the [0 m, 10 m] domain (leading to the grid spacing \(\Delta x = 0.05\) m), and the initial conditions are given as:

• \((p, \alpha_g, u_k, T_k)_L = (10^5\, \text{Pa}, 1-\varepsilon, 100\, \text{m/s}, 300\, \text{K})\) for \(x \leq 5\) m

• \((p, \alpha_g, u_k, T_k)_R = (10^5\, \text{Pa}, \varepsilon, 100\, \text{m/s}, 300\, \text{K})\) for \(x > 5\) m

where \(k = g, l\), and \(\varepsilon = 1.0 \times 10^{-7}\) (\(\varepsilon_{\text{min}} = 1.0 \times 10^{-8}, \varepsilon_{\text{max}} = 1.0 \times 10^{-4}\)). This condition is known to be tough for preserving a constant pressure, say within an \(O(10^{-3})\) error [44]. The computations are conducted with \(\Delta t = 6.0 \times 10^{-6}\) s (CFL = 0.2), up to 0.03 s (5,000 steps).
The results are shown in Fig. 4. Both H-AUSM$^+$-up and H-AUSM$^+$-up (2), as well as G-AUSM$^+$-up and AUSM$^+$-up, showed excellent performance in smooth transition of the two phases (Fig. 4(a)) and in preserving a pressure equilibrium across the contact discontinuity (Fig. 4(b)) within the disturbance of $O(10^{-6})$ downstream the interface, being negligible (i.e., $O(10^{-11})$ smaller) compared with the initial uniform pressure of $10^5$ Pa.

Here the cost comparison is also made, and the results are included in Table 2. Note that the required computational time varies for G-AUSM$^+$-up depending on the tolerance prescribed for Eq. (14g): If it is set as small as $10^{-14}$, the Godunov iteration did not converge in acceptable time, and after the specified iteration count (200,000 iterations in this case) the computation was forced to proceed to the next step (nevertheless, the overall computational time is around two hours); but if the tolerance is as big as $10^{-4}$, the required time (8.2 seconds) was comparable to that of AUSM$^+$-up (8.2 seconds, again). The similar trends will hold for the shocktube and the water-shock/air-bubble interactions in Secs. 3.2.3 and 3.4, respectively. Either H-AUSM$^+$-up or H-AUSM$^+$-up (2) is free from such a user-specified parameter dependence, and showed only a fraction of cost increase (in total 9.9 or 10.4 seconds, respectively) compared with AUSM$^+$-up. Those features of H-AUSM$^+$-up or H-AUSM$^+$-up (2) will be more beneficial when more complex EOSs are employed (such as Jones-Wilkins-Lee (JWL) EOS and Mie-Grüneisen EOS, as done in [28]), in which the Godunov counterpart would invoke many iterations.

3.2 Shock Tube Problems

3.2.1 Air-to-Water Shock Tube

In this test, a rarefaction wave, a phase contact, and a shock are involved. As in the moving contact discontinuity problem, a 1D domain $[0 \text{ m}, 10 \text{ m}]$ is separated by left and right states at $x = 5 \text{ m}$, but with the following different conditions:

- $(\rho, \alpha_g, u_k, T_k)_L = (10^9 \text{ Pa}, 1-\varepsilon, 0 \text{ m/s}, 308.15 \text{ K})$ for $x \leq 5 \text{ m}$
- $(\rho, \alpha_g, u_k, T_k)_R = (10^5 \text{ Pa}, \varepsilon, 0 \text{ m/s}, 308.15 \text{ K})$ for $x > 5 \text{ m}$

where $k = g, l$, and $\varepsilon = 1.0 \times 10^{-7}$ ($\varepsilon_{\text{min}} = 1.0 \times 10^{-8}$, $\varepsilon_{\text{max}} = 1.0 \times 10^{-4}$). The density ratio is $\rho_g/\rho_l \approx 10^3/(1.1 \times 10^4) \approx 0.1$. A grid consisting of the following uniform cells with time step is used:

- 500 cells: $\Delta x = 0.02 \text{ m}$, $\Delta t = 2.0 \times 10^{-6} \text{ s}$ (CFL $\approx 0.2$), computed up to $2.0 \times 10^{-3} \text{ s}$ (1,000 steps)

The results are shown in Fig. 5. As seen in Figs. 5(a) and 5(b), H-AUSM$^+$-up and H-AUSM$^+$-up (2) captured all the important physics: a rarefaction wave in air ($x \approx 4.5 \text{ m}$), a phase interface between air and water ($x \approx 5.5 \text{ m}$), and a shock in water ($x \approx 8.5 \text{ m}$). In the magnified view of the top of the shock front in Fig. 5(c), very slight differences
among the results are observed, but all of them are free from spurious oscillations or kinks found in some fluxes [22, 23].

3.2.2 Water-to-Air Shock Tube

The grid system same to the one used in the air-to-water shock tube is employed, along with the following setup (the density ratio $\rho_l/\rho_g$ is 18):

- $(p, \alpha_g, u_l, T_l)_L = (1 \times 10^7 \text{ Pa}, \varepsilon, 0 \text{ m/s}, 308.15 \text{ K})$ for $x \leq 5 \text{ m}$
- $(p, \alpha_g, u_l, T_l)_R = (5 \times 10^6 \text{ Pa}, 1 - \varepsilon, 0 \text{ m/s}, 308.15 \text{ K})$ for $x > 5 \text{ m}$

and the grid and time step are:

- 500 cells: $\Delta x = 0.02 \text{ m}, \Delta t = 2.0 \times 10^{-6} \text{ s}$ ($\text{CFL} \approx 0.2$), computed up to $2.0 \times 10^{-3} \text{ s}$ (1,000 steps)

The H-AUSM$^+$-up and H-AUSM$^+$-up (2) results are shown in Figs. 6(a) and 6(b). These methods successfully computed this test, and the differences are seen only from the blow-up views in Figs. 6(c) and (d) at a water rarefaction wave and an air shock, respectively. It is noticed that H-AUSM$^+$-up and H-AUSM$^+$-up (2) as well as G-AUSM$^+$-up (2), are free from under/overshoots, whereas AUSM$^+$-up showed slight undershoot at the foot of the rarefaction wave in Fig. 6(c).

3.2.3 Water-to-Air Shock Tube with High Pressure Ratio (PR=1,000)

The same grid system but with higher pressure ratio, $PR = 1,000$, is used. The setup is as follows:

- $(p, \alpha_g, u_l, T_l)_L = (10^8 \text{ Pa}, \varepsilon, 0 \text{ m/s}, 308.15 \text{ K})$ for $x \leq 5 \text{ m}$
- $(p, \alpha_g, u_l, T_l)_R = (10^5 \text{ Pa}, 1 - \varepsilon, 0 \text{ m/s}, 308.15 \text{ K})$ for $x > 5 \text{ m}$

where $k = g, l$, and $\varepsilon = 1.0 \times 10^{-5}$ ($\varepsilon_{\min} = 1.0 \times 10^{-7}, \varepsilon_{\max} = 1.0 \times 10^{-3}$). The density ratio $\rho_l/\rho_g$ is $10^3$.

With this high $PR$, the AUSM$^+$-up standalone had failed because of the severe pressure drop in the (left) water phase at the beginning of computation, as already reported in [17, 23]. H-AUSM$^+$-up and H-AUSM$^+$-up (2), however, as well as G-AUSM$^+$-up, were able to smooth out such a pressure decrease (Figs. 7a, 7b), obviously with the help of a Riemann solver. They, again, showed no under/overshoots (Fig. 7c). Thus, for such a high $PR$, either of Riemann solvers must be used, and the simple HLLC is fine enough. In addition, the cost comparison for this problem (Table 2) showed a similar trend as observed in the moving contact cases in Sec. 3.1.

3.3 Air-Shock/Water-Column Interaction

In this popular 2D benchmark test (examples include [45], [46] and references therein for both simulations and experiments), a shock in air impacting a water column (i.e., 2D droplet) is simulated. 400×200 isotropic cells are
used for a domain of \([-5 \text{ mm}, 5 \text{ mm}] \times [0 \text{ mm}, 5 \text{ mm}]\) to cover the 6.4-mm-diameter water column with its center at the origin (i.e., the diameter being 256 times grid spacing \(\Delta x_{\text{min}} = \Delta y_{\text{min}} = 0.025 \text{ mm}\) in this region); then the cells are stretched toward outer boundaries so that a domain of \([-15 \text{ mm}, 20 \text{ mm}] \times [0 \text{ mm}, 15 \text{ mm}]\) is filled with 900×300 cells in total. Note that this grid system was generated for the purpose of resolving only early stages of evolution of large-scale structures. We used the same grid used in [23] for ease of comparison. Halving the grid resolution destabilized some cases [23], whereas it stabilized some other cases [33].

The initial conditions are the same as in [22]:

\[
\begin{align*}
(p, \alpha_g, u_k, v_k, T_k)_L &= (2.35438 \times 10^5 \text{ Pa}, \varepsilon, 225.86 \text{ m/s}, 0 \text{ m/s}, 381.85 \text{ K}) \text{ for } x \leq -4 \text{ mm} \\
(p, \alpha_g, u_k, v_k, T_k)_R &= (1 \times 10^5 \text{ Pa}, \varepsilon, 0 \text{ m/s}, 0 \text{ m/s}, 293.15 \text{ K}) \text{ for } x > -4 \text{ mm}, \text{ except for } x^2 + y^2 < (3.2 \text{ mm})^2 \text{ where } \\
\alpha_g &= 1 - \varepsilon
\end{align*}
\]

where \(k = g, l\), and \(\varepsilon = 1.0 \times 10^{-5} (\varepsilon_{\text{min}} = 1.0 \times 10^{-5}, \varepsilon_{\text{max}} = 1.0 \times 10^{-4})\). Then the shock starts to move with \(M_{sh} = 1.47\) at \(t = 0\), and hits the water-column at \(t \approx 1.5 \mu\text{s}\). The computations are carried out with \(\Delta t = 1.25 \times 10^{-9} \text{ s} (\text{CFL} \approx 0.15)\) up to \(t = 18.75 \mu\text{s}\) (15,000 steps) for H-AUSM \(^+\)-up and G-AUSM \(^+\)-up, and to \(6.25 \mu\text{s}\) (5,000 steps) for AUSM \(^+\)-up when the phase interface begins to deform. H-AUSM \(^+\)-up (2) computation unfortunately diverged at the impact of the shock on the water-column (around 1,200 steps; \(t \approx 1.5 \mu\text{s}\)). This is a natural consequence, remembering the fact that one method closer to the exact solver (e.g., G-AUSM \(^+\)-up) is more robust and accurate than its approximate counterpart (e.g., AUSM \(^+\)-up) [23] – in this case, H-AUSM \(^+\)-up belongs to the former, while H-AUSM \(^+\)-up(2) the latter.

To specify a smooth distribution of void fraction at the interface of the circular water column on a rectangular (Cartesian) grid, it is necessary to create a transition region of certain width, \(\pm 2\Delta x_{\text{min}}\) in this study, about the interface; otherwise, the computation destabilized after the initial representation of zigzagged interface. The same formula used for the “vanishing” phase treatment again is applied here.

\[
(\alpha_g)_{\text{adjust}} = G(\xi_2) \cdot \varepsilon + \left(1 - G(\xi_2)\right) \cdot \left(1 - \varepsilon\right),
\]

\[
G(\xi_2) = -\xi_2^2 \left(2\xi_2^2 - 3\right),
\]

\[
\xi_2 = \sqrt{\frac{x^2 + y^2 - (r - 2\Delta x_{\text{min}})}{4\Delta x_{\text{min}}}}, \quad r - 2\Delta x_{\text{min}} \leq \sqrt{x^2 + y^2} \leq r + 2\Delta x_{\text{min}}, \quad r = 3.2 \text{ mm}.
\]
At the bottom boundary, the conventional slip condition is imposed; that is, only the \( y \)-component velocity is reflected, and the other variables are simply extrapolated from the interior cells. Treatments at the other boundaries are typical: the left boundary is the inlet condition, the right is the outlet, and the top boundary is the side. Those far-field boundaries are far enough away from the water column to influence the flow of interest, and variables there are fixed in time [23].

The results of H-AUSM\(^+\)-up are shown in Fig. 8, in which numerical Schlieren function \( (1 + \alpha^2) \log(|N\rho| + 1) \) [17] is used with the range between 4 and 20 along with pressure between 10,000 and 50,000 Pa. Key flow structures are smoothly captured, such as a growing, diffracted shock outside the water-column, and shock reflections inside the column (detailed explanations are found in [17, 23]). Compared with the G-AUSM\(^+\)-up results in Fig. 9, no remarkable difference is found. The more efficient AUSM\(^+\)-up, however, showed wiggles at the phase interface (Fig. 10).

It is therefore confirmed that inclusion of the Riemann solver at phase interfaces is critical in computing this 2D, challenging, shock/water-column interaction problem, and that HLLC is robust enough if implemented only to find middle region velocity and pressure; even if the iterative Godunov solver is used instead, almost the same solution is obtained; if no Riemann solver is called, the solution is degenerated.

We would like to mention that, whereas the present setup was designed for early stages of this problem, the HLLC combined with THINC sharpening successfully resolved key features for long computations in our recent work [33].

### 3.4 Water-Shock/Air-Bubble Interaction

This final test is opposite to the previous test, with a water shock impacting a column of an air (2D bubble), but with a much higher pressure ratio \((PR \approx 1.6 \times 10^4)\). The grid system is the same as that in the shock/water-column interaction problem. The initial conditions are the same as in [17, 23]:

- \((p, \alpha_g, u_k, v_k, T_k)_L = (1.6 \times 10^9 \text{ Pa}, 1 - \varepsilon, 661.81 \text{ m/s}, 0 \text{ m/s}, 595.13 \text{ K})\) for \(x \leq -4 \text{ mm}\)
- \((p, \alpha_g, u_k, v_k, T_k)_R = (1.01325 \times 10^5 \text{ Pa}, 1 - \varepsilon, 0 \text{ m/s}, 0 \text{ m/s}, 292.98 \text{ K})\) for \(x > -4 \text{ mm}\), except for \(x^2 + y^2 < 3.2 \text{ mm}^2\) where \(\alpha_g = \varepsilon = 1.0 \times 10^{-3}\) (\(\varepsilon_{\min} = 1.0 \times 10^{-3}, \varepsilon_{\max} = 1.0 \times 10^{-1}\))

The shock moves at \(M_{sh} = 1.51\) at \(t = 0\) and hits the air bubble at \(t \approx 0.3 \mu\text{s}\). The computations are carried out with \(\Delta t = 3.125 \times 10^{-10} \text{ s (CFL} \approx 0.05)\) up to 5.0 \(\mu\text{s (16,000 steps)}. For a stability reason, only in this case, the threshold for
effective length is $5\varepsilon$, and the speed of sound is defined individually for each phase. This slight change may confuse some readers, but this value is in consistent with the earlier work [23] for comparison.

The results of H-AUSM'-up, H-AUSM'-up (2), and G-AUSM'-up are shown in Figs 11, 12, and 13 (in which numerical Schlieren function $\log \{ |V \rho| + 1 \} \) [17, 23] is used with the range between 8 and 14 and with pressure between $10^8$ and $2 \times 10^9$ Pa, respectively; AUSM'-up blew up immediately after the shock hit the bubble. The results of H-AUSM'-up and G-AUSM'-up, again, agree with each other. As time progresses in both cases, the left rim of the air bubble is pushed forward (to the right), and it is deformed and compressed into a smaller volume (Figs. 11(a)–(d), 12(a)–(b), and 13(a)–(b)). After the left rim of the bubble reaches the right rim (then H-AUSM'-up (2) diverged, as in the last test case), the pressure there begins to exceed the water ambient pressure (up to $2.9 \times 10^9$ Pa at 3.75 $\mu$s, Figs. 11(e), (f)), thereby pushing the surrounding fluids away. Subsequently, the bubble is broken into two separate bubbles, which are still resolved well by the current grid (5.0 $\mu$s, Figs. 11(g), (h) and 13(c), (d)). Further details are explained by Chang and Liou in [17].

As in the shock/water-column case, it is again confirmed that the HLLC Riemann solver is both robust and simple at phase interfaces in computing this shock/air-bubble interaction problem, if used only to find middle region velocity and pressure; the Godunov solver is expensive if the convergence criterion is strict (Table 2), and AUSM'-up standalone is unstable. The results of all the tests are summarized in Tables 2 for reference.

Now that the HLLC-extended AUSM is proved to be a well-balanced choice with respect to simplicity and robustness, our immediate target is to enhance the sharpness of phase interfaces [33] using THINC method [31, 32] within the present framework‡.

4. Conclusions

The two-fluid, AUSM'-up flux function coupled with Godunov exact Riemann solver has been successfully extended to AUSM'-up/HLLC Riemann solver combination.

It is confirmed that inclusion of Riemann solver at phase interfaces is critical in computing challenging problems involving high pressure ratio, and that HLLC is robust enough if implemented only to find middle region velocity and pressure; if implemented to compute all the variables by fully following the HLLC procedure, its robustness is weakened; even if the iterative Godunov solver is used instead (AUSM'-up/Godunov), almost the same solution is

‡ Although Ref. [33] is already published at the time of publication of this work, Ref. [33] is partly based on the present work in the point that HLLC is employed for initial guess of the Godunov solver, as a variant of the present strategy.
obtained, but its cost heavily depends on the prescribed convergence criterion; if no Riemann solver is called (AUSM’-up only), the solution accuracy and/or robustness is degraded.

Acknowledgments

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### Table 1. Summary of flux evaluation of schemes tested in this note.

<table>
<thead>
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<th>Methods</th>
<th>Gas-liquid interface</th>
<th>Gas-gas interface</th>
<th>Liquid-liquid interface</th>
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<td>AUSM-up</td>
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<td>AUSM (Eqs. 12 and 13)</td>
<td>AUSM (Eqs. 12 and 13)</td>
</tr>
<tr>
<td>G-AUSM-up</td>
<td>Godunov (Eqs. 14)</td>
<td>AUSM (Eqs. 12 and 13)</td>
<td>AUSM (Eqs. 12 and 13)</td>
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<tr>
<td>H-AUSM-up</td>
<td>HLLC (Eqs. 14(j)-14(s), and 15)</td>
<td>AUSM (Eqs. 12 and 13)</td>
<td>AUSM (Eqs. 12 and 13)</td>
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<tr>
<td>H-AUSM-up (2)</td>
<td>HLLC(2) (Eqs. 14(q)-14(s), 15 and 16)</td>
<td>AUSM (Eqs. 12 and 13)</td>
<td>AUSM (Eqs. 12 and 13)</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Methods</th>
<th>Moving phase contact discontinuity</th>
<th>Air-to-water shock tube</th>
<th>Water-to-air shock tube</th>
<th>Water-to-air shock tube with high PR</th>
<th>Shock/water-column interaction</th>
<th>Shock/air-bubble interaction</th>
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</thead>
<tbody>
<tr>
<td>AUSM-up</td>
<td>S (8.2s)</td>
<td>S</td>
<td>F</td>
<td>A</td>
<td>F</td>
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<tr>
<td>G-AUSM-up</td>
<td>S (8.2s – 7210s)</td>
<td>S</td>
<td>S (2.8s – 761s)</td>
<td>S</td>
<td>S (2.1s – 680s / step)</td>
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<tr>
<td>H-AUSM-up</td>
<td>S (9.9s)</td>
<td>S</td>
<td>S (3.4s)</td>
<td>S</td>
<td>S (2.3s/ step)</td>
<td></td>
</tr>
<tr>
<td>H-AUSM-up (2)</td>
<td>S (10.4s)</td>
<td>S</td>
<td>S (3.4s)</td>
<td>F</td>
<td>A (2.3s/ step)</td>
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